

Pseudo-Spin Symmetry and its Applications¹

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Abstract. The pseudo-spin symmetry is reviewed. A mapping that produces the separation of the total angular momentum into pseudo-orbital and pseudo-spin degrees of freedom is discussed, together with the analytic transformations that take us from the normal parity space to the eigenstates of a pseudo-oscillator with one quanta less. The many-particle version of the unitary transformation to the pseudo-SU(3) space is established. As an example, these symmetries are used to describe the double beta decay phenomenon in heavy deformed nuclei.

INTRODUCTION

The pseudo-spin symmetry was introduced in [1,2], where a separation of the single particle Hilbert space into pseudo orbital and pseudo-spin degrees of freedom was proposed. This new coupling scheme provides a simple interpretation of many striking features of the Nilsson orbits: i) The observed approximate degeneracy of the single particle levels of the type $(N, l, s)j$ and $(N, l + 2, s)j + 1$ for the spherical case or the one associated to the levels with asymptotic quantum numbers $[N, N_z, \Lambda] \Omega = \Lambda + 1/2$ and $[N, N_z, \Lambda + 2] \Omega = \Lambda + 3/2$ for the large deformation case. These energy orbitals can be considered as pseudo-spin orbit doublets, implying that the strength of the pseudo spin-orbit interaction is small. ii) The expectation value of the pseudo-spin operator, iii) the calculation of the decoupling parameters, and iv) the matrix elements of the Coriolis interaction [1]. More recently the pseudo-spin symmetry has been considered to be useful to characterize identical bands [3] and as a signature of superdeformation [4].

For heavy nuclei the nuclear shells can be divided into two parts: the associated to the single particle orbits of the same parity, which are called normal

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parity levels and the intruder that comes from the shell above of opposite parity and thus it has been called unique or abnormal parity orbital. Thus for a given shell, N_α one has $j_\alpha^N = \{1/2, 3/2, \dots, N_\alpha - 1/2\}$, $j_\alpha^A = N_\alpha + 3/2$, where $\alpha = \pi$ or ν for protons and neutrons, respectively. This separation of the space for the single particle states of Nilsson or Wood Saxon potentials is reasonable because it has been proved that the intruder orbitals are nearly unmixed for standard deformations [5]. When relabelled in the pseudo spin scheme, the normal parity levels form a major shell for a pseudo-oscillator potential with $\tilde{N}_\alpha = N_\alpha - 1$ quanta. Since the symmetry of this oscillator is of course SU(3), one may consider the pseudo-SU(3) coupling scheme.

In the Second Section, we review the transformation to the pseudo space introduced by Bohr, Mottelson and Hamamoto (BHM) [6] and the mappings from oscillators to pseudo-oscillators wave functions for the spherical and asymptotic cases [7]. We emphasize the difference between the pseudo-spin and the pseudo-oscillator symmetries. The first one implies the separation of the normal parity degrees of freedom into pseudo spin and pseudo orbital spaces (called pseudo-space) while the second one for the one particle case is related with the use of oscillator wavefunctions with spin to describe the pseudo-space. At the end of this section, the many-particle unitary transformation to the pseudo-SU(3) space is introduced and applied to one body operators. In the final section, theoretical calculations for the 2ν double beta decays of ^{150}Nd and ^{238}U are presented [8].

UNITARY TRANSFORMATIONS

The BHM unitary transformation acts on the angular-spin parts of the wave functions of the particle, without affecting the radial motion. It is given by the scalar product between the spin operator \vec{S} and \hat{r} a unit vector in the direction of the position of the particle, $U_{BHM} = 2i\hat{r} \cdot \vec{S}$. It is straightforward, to calculate the action on the single-particle wave function of the three dimensional harmonic oscillator with quantum numbers

$$\psi_{N(l,s)jm}(\vec{r}) = R_{Nl}(r) \sum_{\mu\sigma} \langle l\mu, s\sigma | jm \rangle Y_{l\mu}(\theta, \phi) \chi_{s\sigma}, \quad (1)$$

which can be written as a product of the radial part times the superposition of two spinor wave functions [6]. This transformation gives rise to the results $l = j \pm 1/2 \mapsto \tilde{l} = j \mp 1/2$. Thus it produces the breaking of the normal degrees of freedom into pseudo-spin and pseudo-orbital spaces, but it does not give the wave functions of a pseudo-oscillator with one quanta less.

In the set of wave functions (2) corresponding to a given N value, we disregard those with $j = N + \frac{1}{2}$, which define the unique or abnormal space, and only consider the remaining wave functions of the normal parity orbitals. These

states can be mapped onto the eigenstates $|\tilde{N}(\tilde{l}\tilde{s})\tilde{j}\tilde{m}\rangle$ of a pseudo-oscillator via the following unitary operator [7]

$$U = 2(\vec{\xi} \cdot \vec{S})(\hat{N} - 2\vec{L} \cdot \vec{S})^{-\frac{1}{2}}, \quad (2)$$

where $\vec{\xi}$ is the annihilation harmonic oscillator quantum. The action of U onto eigenstates of a three dimensional harmonic oscillator gives the result

$$U|N(l, s)jm\rangle = |\tilde{N}(\tilde{l}\tilde{s})\tilde{j}\tilde{m}\rangle, \quad (3)$$

with the relations between the labels $\tilde{N} = N - 1$, $\tilde{s} = s$, $\tilde{j} = j$, $\tilde{m} = m$, and $\tilde{l} = l \pm 1$ according to whether $j = l \pm 1/2$. Although the previous mapping connecting the normal parity harmonic oscillator eigenstates with the full set of eigenstates of a pseudo-oscillator was proposed long ago [1, 2], the explicit form of the unitary operator (3) was constructed for the first time in [7]. More recently, the transformation to the pseudo-oscillator space has been carried out by using the Algebraic Generator Coordinate Method [9].

Now we consider the asymptotic wave functions of the Nilsson Hamiltonian [7] associated with large deformations. Then the Nilsson orbitals are described by the states characterized by $|[N, N_z, \Lambda]\Omega\rangle$, the cylindrical harmonic oscillator states with spin. The degeneracy mentioned above can be explained through the unitary transformation

$$U_\infty = 2(\xi_+ S_+ + \xi_- S_-)(\hat{N}_\rho - 2\vec{L}_z \cdot \vec{S}_z)^{-\frac{1}{2}}, \quad (4)$$

where the operators ξ_\pm and η_\pm are the spherical components of the creation $\vec{\eta}$ and annihilation $\vec{\xi}$ harmonic oscillator operators, the spin operators are expressed in terms of the Pauli matrices $S_m = \sigma_m/2$ and \hat{N}_ρ is the number operator in the plane.

The U_∞ and their corresponding hermitean conjugated operator U_∞^\dagger are unitary, if they act onto states belonging to the normal space orbitals, that is if we disregard the levels of a given N shell that satisfy $\Omega = N_\rho + 1/2$. Thus, the unitary transformations for the spherical and asymptotic Nilsson orbitals to the pseudo-oscillator wave functions are not equivalent. This fact which has not been previously emphasized, is reflected in the separation itself into normal and abnormal parity spaces. However for standard deformations the exact wave functions of the Nilsson Hamiltonian can be evaluated and it is observed that there is no mixing of the unique (spherical) parity orbital with the remaining levels of the shell [10], which suggest that for these deformations the spherical transformation is the most appropriate. The division of the shell model space into normal and unique as proposed in the asymptotic limit may be useful in applications to superdeformation phenomena.

For a system of n particles the unitary transformation to the pseudo-oscillator space is defined by

$$U = \prod_s^n U_s. \quad (5)$$

In the Fock space, if we denote by \hat{U} the corresponding unitary operator associated to U , then a general one body operator F is mapped to the pseudo-SU(3) space by $\hat{F} = \hat{U} F \hat{U}^\dagger$. If two single particle state vectors $|\alpha\rangle$ and $|\tilde{\alpha}\rangle$ are related by a unitary transformation U (see Eq. 2), then the fermion operators are related by $\hat{U} a_\alpha^\dagger \hat{U}^\dagger = a_{\tilde{\alpha}}^\dagger$. Then, F in the pseudo-space is given by

$$\hat{F} = \sum_{N,l,j,\mu}' \sum_{N',l',j',\mu'}' \langle N'(l', \frac{1}{2}) j' \mu' | F_1 | N(l, \frac{1}{2}) j \mu \rangle a_{\tilde{N}', \tilde{l}', \tilde{j}', \tilde{\mu}'}^\dagger a_{\tilde{N}, \tilde{l}, \tilde{j}, \tilde{\mu}}, \quad (6)$$

where the primes on the sum indicate that the unique parity orbitals are excluded. A similar result is obtained for the two body operators, that is only the labels of the fermion creation and annihilation operators are changed by its pseudo quantum numbers. The previous results give formal support to the recipe indicated in Refs. [1, 11]. The many body expressions in the pseudo space of the $SU(3)$ generators are given by the series expansions

$$\hat{\tilde{L}}_q = k_L \hat{\tilde{L}}_q + \dots, \quad \hat{\tilde{Q}}_\mu = k_Q \hat{\tilde{Q}}_\mu + \dots, \quad (7)$$

where only the leading terms of the series are indicated. These operators $\hat{\tilde{L}}_q, \hat{\tilde{Q}}_\mu$ have the same form as the $SU(3)$ generators but of a shell with one quanta less, but we want to emphasize that they are not the transformed operators $\hat{\tilde{L}}_q, \hat{\tilde{Q}}_\mu$ to the pseudo space.

DOUBLE BETA DECAY

Double beta decay is a rare transition between two nuclei with the same mass number A involving change of the nuclear charge number Z by two units. This exotic phenomenon is a useful tool to test the lepton number conservation, neutrino properties and models of nuclear structure [12]. It can be classified into various modes according to the light particles besides the electrons associated with the decay. The two neutrino mode ($\beta\beta_{2\nu}$), in which two electrons and two neutrinos are emitted, takes place independently of the neutrino properties, and conserves the electric charge and lepton number. The 0ν mode violates lepton number conservation and therefore it is forbidden in the standard electroweak theory. To proceed the 0ν decay, the virtual neutrino must be emitted in one vertex and absorbed in the other one, thus it is required that: i) the exchanged neutrino is a Majorana particle ($\nu = \bar{\nu}$) and ii) both neutrinos have a common helicity component [13].

Next, we restrict to describe, within the pseudo-SU(3) formalism, the 2ν double beta decays of ^{150}Nd and ^{238}U . The decay rate of the 2ν -mode can be calculated through the formulae

$$(\tau_{2\nu}^{1/2})^{-1} = G_{2\nu} |M_{2\nu}^{GT}|^2, \quad (8)$$

where $G_{2\nu}$ is a kinematic factor and $M_{2\nu}^{GT} = M_{2\nu}$ is a nuclear matrix element strongly dependent on the considered nuclear model, that is

$$M_{2\nu} = \sum_N \frac{1}{E_0 + E_N - E_i} \langle 0_f^+ || \Gamma || 1_N^+ \rangle \langle 1_N^+ || \Gamma || 0_i^+ \rangle. \quad (9)$$

The Γ is denoting the Gamow-Teller operator $\Gamma_m = \sum_s \sigma_{ms} t_s^-$ and the $E_0 = \frac{1}{2} Q_{\beta\beta} + m_e c^2$ is the half of the total energy released. The E_N gives the energy of the intermediate state $|1_N^+\rangle$. The E_i is the energy of the ground state of the initial nucleus $|0_i^+\rangle$ and the ket $|0_f^+\rangle$ describes the ground state of the final nucleus.

In order to compute $M_{2\nu}$, we have to perform a sum over all the intermediate states. Fortunately, an alternative form of calculate this matrix element has been developed [14], *i.e.*,

$$M_{2\nu} = \frac{1}{E_0} \langle 0_f^+ | \sum_m (-1)^m \Gamma_{-m} F_m | 0_i^+ \rangle, \quad (10)$$

where the operator F_m is defined by:

$$F_m = \sum_{\lambda} \frac{(-1)^{\lambda}}{E_0^{\lambda}} [H, [H, \dots, [H, \Gamma_m] \dots]]^{(\lambda - \text{times})}. \quad (11)$$

A reasonable model for describing spectra and $BE2$ transitions of heavy deformed nuclei is [8]:

$$H = \sum_{\alpha} H_{\alpha} - \frac{1}{2} \chi \mathbf{Q}^a \cdot \mathbf{Q}^a + \zeta_1 K^2 + \zeta_2 L^2, \quad (12)$$

where H_{α} denotes the spherical Nilsson Hamiltonian for neutrons or protons plus a constant term V_{α} , which represent the depth of the potential well. The quadrupole-quadrupole interaction in a given shell and K^2 is a linear combination of L^2 , X_3 and X_4 , which are rotational scalar operators built with generators of the algebra of $SU(3)$ [15].

Notice that the quadrupole-quadrupole force, L^2 and the K^2 interaction are independent of the spin degrees of freedom and symmetric in the neutron and proton components. To evaluate (13), we express H_{α} and Γ_m in the second quantization formalism,

$$H_{\alpha} = \hbar\omega \sum_{\eta l j m} \epsilon_{\alpha}(\eta, l, j) a_{\eta l \frac{1}{2} j m \alpha}^{\dagger} a_{\eta l \frac{1}{2} j m \alpha}, \quad (13)$$

$$\Gamma_m = \sum_{\pi \nu} \sigma(\pi, \nu) A(\pi, \nu, m), \quad (14)$$

with $A(\pi, \nu, m) = [a_{\eta_{\pi} l_{\pi} \frac{1}{2}; j_{\pi}}^{\dagger} \otimes \tilde{a}_{\eta_{\nu} l_{\nu} \frac{1}{2}; j_{\nu}}]_m^1$ denoting the angular coupling of proton creation and neutron annihilation operators, ϵ_{α} the single particle energies

and

$$\sigma(\pi, \nu) \equiv \sum_{\pi \nu} \sqrt{\frac{2j_\pi + 1}{3}} \langle \eta_\pi l_\pi \frac{1}{2}; j_\pi || \sigma || \eta_\nu l_\nu \frac{1}{2}; j_\nu \rangle. \quad (15)$$

Afterwards some algebraic manipulations, $M_{2\nu}$ is rewritten by

$$M_{2\nu} = \sum_{\pi \nu} \sum_{\pi' \nu'} \sigma(\pi', \nu') \sigma(\pi, \nu) \frac{\langle 0_f^+ | \vec{A}(\pi', \nu') \cdot \vec{A}(\pi, \nu) | 0_i^+ \rangle}{E_0 + \epsilon(\eta_\pi, l_\pi, j_\pi) - \epsilon(\eta_\nu, l_\nu, j_\nu)} \quad (16)$$

In the pseudo SU(3) Hilbert space the above sum is restricted to $\eta_\pi = \eta_\nu$, $l_\pi = l_\nu$, $\eta_{\pi'} = \eta_{\nu'}$, $l_{\pi'} = l_{\nu'}$, $j_\pi = j_\nu + 1$. Following [8], the single particle energy difference in the denominator takes the form

$$\epsilon(\eta, l, j_\pi) - \epsilon(\eta, l, j_\nu) = -\hbar\omega k_\pi 2j_\pi + \Delta_C. \quad (17)$$

The constants k_α are well known [16] and Δ_C is used to determine the value $V_\nu - V_\pi$. The Δ_C is the difference Coulomb energy and it is evaluated by the expression

$$\Delta_C = \frac{0.70}{A^{1/3}} [2Z + 1 - 0.76((Z + 1)^{4/3} - Z^{4/3})] \text{MeV}. \quad (18)$$

The description of the correlated deformed ground states is done using the pseudo SU(3) scheme for the normal parity space and seniority zero configurations, that is all the nucleons coupled by pairs to angular momentum zero, for the unique part [11, 15]. The occupancies of these spaces are determined from the corresponding Nilsson diagrams by selecting a reasonable deformation and filling each level with a pair of particles in order of increasing energy. These numbers fix the totally antisymmetric irreducible representations (irreps) of the unitary groups associated to the normal $U((N_\alpha + 1)(N_\alpha + 2))$ and unique $U(2N_\alpha + 4)$ spaces.

For the normal parity space, one separates the degrees of freedom in pseudo-orbital $U(\Omega_\alpha^N)$ and pseudo-spin $U_\alpha(2)$ parts, with $\Omega_\alpha^N = (N_\alpha + 1)(N_\alpha + 2)/2$ and their irreps $\{\tilde{f}_\alpha\}$ are characterized by the partitions of the number of particles in the normal part, n_α^N . Then of the pseudo $SU(3)$ irreps, $(\lambda_\alpha, \mu_\alpha)$, contained in $\{\tilde{f}_\alpha\}$, one considers those with maximum eigenvalue of the Casimir operator, $(C_2)_\alpha = (\lambda_\alpha + \mu_\alpha + 3)(\lambda_\alpha + \mu_\alpha) - \lambda_\alpha \mu_\alpha$. In Table 1, the results found for the participant nuclei in the double beta decays of ^{150}Nd and ^{238}U are presented.

Table 1. Ground states in the Pseudo-SU(3) coupling scheme.

NUCLEUS	n_π^N	n_π^A	n_ν^N	n_ν^A	$U(\Omega_\pi^N)$	$U(\Omega_\nu^N)$	$SU_\pi(3)$	$SU_\nu(3)$
^{150}Nd	6	4	6	2	$\{2^3\}$	$\{2^3\}$	(12, 0)	(18, 0)
^{150}Sm	6	6	4	2	$\{2^3\}$	$\{2^2\}$	(12, 0)	(12, 2)
^{238}U	6	4	12	8	$\{2^3\}$	$\{2^6\}$	(18, 0)	(36, 0)
^{238}Pu	6	6	10	8	$\{2^3\}$	$\{2^5\}$	(18, 0)	(30, 4)

Table 2. Theoretical estimates for the half-life $\beta\beta$ -decay in the 2ν mode.

<i>Transition</i>	$< 0_f^+ \Gamma^2 0_i^+ >$	$E[MeV]$	$\tau_{theo}^{1/2}[y]$	$\tau_{exp}^{1/2}[y][13]$
$^{150}Nd \rightarrow ^{150}Sm$	1.31	12.2	6.0×10^{18}	$9-17 \times 10^{18}$
$^{238}U \rightarrow ^{238}Pu$	1.51	16.8	1.4×10^{21}	2×10^{21}

Finally we used the strong coupled limit [11], and from the Kronecker product $(\lambda_\pi, \mu_\pi) \times (\lambda_\nu, \mu_\nu)$, the $(\lambda_\pi + \lambda_\nu, \mu_\pi + \mu_\nu)$ irrep will dominate the low-lying energy structure.

In Table 2, we display the calculated Gamow-Teller matrix elements, energy denominators, predicted and experimentally determined double beta half lives of ^{150}Nd and ^{238}U . They are given in the last column of Table 2 and these half lives are lower bounds because the nuclear matrix elements could be reduced by considering, for example the inclusion of the pairing interaction, and therefore giving longer (never shorter) $\beta\beta$ half lives.

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